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Ideal Completion of Join Semilattice over T-algebra

西澤　弘毅*・古澤　仁†

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和文要旨：この論文は、クウォンテールの圏からべき等半環の圏への忘却関手の左随伴を明示的に構成する方法を二つ与える。一つの方法は、イデアル完備化が、完備上半束の圏から上半束の圏への忘却関手の左随伴を与えることを示す方法である。この方法に対して、上半束からＴ代数上の上半束へ一般化すると、例として、クウォンテールの圏からべき等半環の圏への忘却関手の左随伴を扱えるようになる。もう一つの方法は、べき等半環のイデアル完備化が忘却関手の左随伴を与えることを、直接的に証明する方法である。

【キーワード】べき等半環、クウォンテール、イデアル完備化、分配律

Abstract：This paper provides an explicit construction of the left adjoint to the forgetful functor from the category of quantales to the category of idempotent semirings in two ways. The first way shows that an ideal completion gives the left adjoint to the forgetful functor from the category of complete join semilattices to the category of join semilattices. Generalizing join semilattices to join semilattices over *T*-algebras, the left adjoint to the forgetful functor from the category of quantales to the category of idempotent semirings is given as an example. The second way is to directly prove that the ideal completion of idempotent semirings gives the left adjoint.

【Keywords】idempotent semiring, quantale, ideal completion, distributive law

1　Introduction

This paper provides an explicit construction of the left adjoint to the forgetful functor from the category of complete join semilattices to the category of join semilattices. Similarly, one can construct the left adjoint to the forgetful functor from the category of quantales to the category of idempotent semirings. A quantale is a complete join semilattice together with a monoid structure whose associative multiplication distributes over arbitrary joins. An idempotent semiring is a join semilattice together with a monoid structure whose associative multiplication distributes over finite joins. Both of these left adjoints are defined by an ideal completion. In order to show why both of these left adjoints are defined by the ideal completion, we generalize quantales and idempotent semirings to complete join semilattices over *T*-algebras and join semilattices over *T*-algebras, respectively. By generalization, the left adjoint to the forgetful functor from the category of quantales to the category of idempotent semirings is given as an example. We also give a direct proof for the case of the category of quantales and the category of idempotent semirings.

This paper is organized as follows: Section 2 defines monads for join semilattices. Section 3 shows that a left ad-

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joint between categories of algebras is defined by absolute coequalizer construction. Section 4 shows that the ideal completion gives the absolute coequalizer for join semilattices. Section 5 generalizes results in Section 3 for monads combined by distributive laws. Section 6 directly proves that the ideal completion gives the left adjoint to the forgetful functor from the category of quantales to the category of idempotent semirings without notions of monads and distributive laws. Section 7 summarizes this work and discusses future work.

2　Monads for Join Semilattices

Definition 2.1. A *join semilattice* is a tuple $(S, \leq, \vee)$ with a partially ordered set $(S, \leq)$ and the join or the least upper bound $\bigvee A$ for a finite subset A of S.

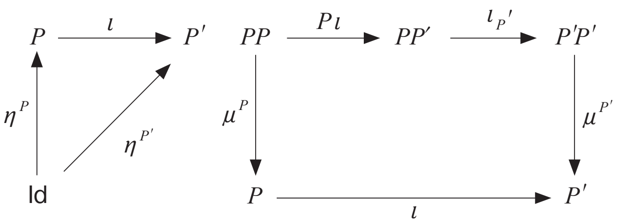
A join semilattice S must have the least element 0 since the empty set $\emptyset$ is a finite subset of S and $\vee \emptyset = 0$.

SLat denotes the category whose objects are join semilattices and whose arrows are homomorphisms between them. SLat is equivalent to the Eilenberg-Moore category $\wp_f$-Alg of the finite powerset monad $\wp_f$, whose endofunctor sends a set X to the set of finite subsets $\wp_f(X)=\{A \subseteq X \mid |A|<\omega\}$, whose unit sends an element x in X to $\{x\}$ in $\wp_f(X)$, and whose multiplication sends a subset family α in $\wp_f(\wp_f(X))$ to its union $\cup\alpha=\{x \mid \exists X\in\alpha, x\in X\}$ in $\wp_f(X)$. The forgetful functor from $\wp_f$-Alg to Set has the left adjoint which sends a set X to $(\wp_f(X), \cup: \wp_f(\wp_f(X)) \to \wp_f(X))$. The counit for $\wp_f$-algebra $(S, \vee)$ is $\vee: (\wp_f(S), \cup) \to (S, \vee)$.

Definition 2.2. A *complete join semilattice* is a tuple $(S, \leq, \vee)$ with a partially ordered set $(S, \leq)$ and the join or the least upper bound $\bigvee A$ for a subset A of S.

We write CSLat for the category whose objects are complete join semilattices and whose arrows are homomorphisms between them. CSLat is equivalent to the Eilenberg-Moore category $\wp$-Alg of the powerset monad $\wp$, whose endofunctor sends a set X to the set of all subsets $\wp(X)=\{A \mid A \subseteq X\}$, whose unit sends an element x in X to $\{x\}$ in $\wp(X)$, and whose multiplication sends a subset family α in $\wp(\wp(X))$ to its union $\cup\alpha=\{x \mid \exists X\in\alpha, x\in X\}$ in $\wp(X)$. The forgetful functor from $\wp$-Alg to Set has the left adjoint which sends a set X to $(\wp(X), \cup: \wp(\wp(X)) \to \wp(X))$. The counit for $\wp$-algebra $(S, \vee)$ is $\vee: (\wp(S), \cup) \to (S, \vee)$.

Definition 2.3. Let $P = (P, \mu^P, \eta^P)$ and $P'=(P', \mu^{P'}, \eta^{P'})$ be monads on C. A *monad map* ι from P to P' is a natural transformation from P to P' satisfying the following diagrams.



Lemma 2.4. *Let $P=(P, \mu^P, \eta^P)$ and $P'=(P', \mu^{P'}, \eta^{P'})$ be monads on C. If ι is a monad map from P to P', then, the following G is a functor from P'-Alg to P-Alg.*

- *For a P'-algebra (c, p'), $G(c, p')=(c, p' \circ \iota_c)$.*
- *For a P'-algebra homomorphism f: $(c_1, p'_1) \to (c_2, p'_2)$, $Gf=f$.*

Proof. $G(c, p')$ *is a* P-algebra by the following equations.

$$p' \circ \iota_c \circ \eta^P_c = p' \circ \eta^{P'}_c = \mathrm{Id}$$

$$p' \circ \iota_c \circ \mu^P_c = p' \circ \mu^{P'}_c \circ \iota_{P'c} \circ P\iota_c = p' \circ P'p' \circ \iota_{P'c} \circ P\iota_c = p' \circ \iota_c \circ Pp' \circ P\iota_c$$

Gf is a P-algebra homomorphism by the following equation.

$$f \circ p'_1 \circ \iota_{c1} = p'_2 \circ P'f \circ \iota_{c1} = p'_2 \circ \iota_{c2} \circ Pf$$

□

We write $-\circ\iota$ for the above functor G.

Example 2.5. Let P be the finite powerset monad $\wp_f$. Let P' be the powerset monad $\wp$. Let ι_X be the inclusion from $\wp_f(X)$ to $\wp(X)$. Then, ι is a monad map from $\wp_f$ to $\wp$. The functor $-\circ\iota$ is the forgetful functor from CSLat to SLat.

3 Left Adjoint by Absolute Coequalizers

The next theorem is a corollary of the Theorem 2(b) of Section 3.7 of the book [2].

Theorem 3.1. *Let C, D, D' be categories and G, U, U' be functors satisfying the following conditions.*

- *G: $D' \to D$ is a functor.*
- *A functor U: $D \to C$ has a left adjoint F (call its unit η and its counit ε).*
- *A functor U': $D' \to C$ has a left adjoint F' (call its unit η' and its counit ε').*
- *$U \circ G$ is natural isomorphic to U'.*
- *U is monadic.*
- *For an object d in D, the parallel pair*

$$F'UFUd \xrightarrow{F'U\varepsilon_d} F'Ud$$

and

$$F'UFUd \xrightarrow{F'UF\eta'_{Ud}} F'UFU'F'Ud \xrightarrow{\cong\circ F'U\varepsilon_{GF'Ud}\circ\cong} F'U'F'Ud \xrightarrow{\varepsilon'_{F'Ud}} F'Ud$$

has a coequalizer in D'.

Then, G has a left adjoint which sends an object d in D to the codomain of the above coequalizer.

Theorem 3.2 (Corollary of Beck's theorem [6]). *If a functor U': $D' \to C$ is monadic, then U' creates coequalizers of those parallel pairs f,g in D' for which $U'f$, $U'g$ has an absolute coequalizer in C.*

Example 3.3. The forgetful functor U' from CSLat to Set is monadic. Therefore, it creates coequalizers of those parallel pairs f, g in CSLat for which $U'f$, $U'g$ has an absolute coequalizer in Set.

Theorem 3.4. *Let $P=(P, \mu^P, \eta^P)$ and $P'=(P', \mu^{P'}, \eta^{P'})$ be monads on C. Let ι be a monad map from P to P'. For a P-algebra (c, p), let $e_{(c,p)}$: $P'c \to E(c, p)$ be an absolute coequalizer of $P'p$ and $\mu^{P'}_c \circ P'\iota_c$ in C.*

$$P'Pc \underset{\mu^{P'}_c\circ P'\iota_c}{\overset{P'p}{\rightrightarrows}} P'c \xrightarrow{e_{(c,p)}} E(c,p)$$

Then, the functor $-\circ\iota$: P'-Alg $\to$ P-Alg has a left adjoint L, where $L(c, p)$ is the P'-algebra on $E(c, p)$ created by the forgetful functor from P'-Alg to C.

Proof. Let D=P-Alg and D'=P'-Alg in Theorem 3.1. Let U be the forgetful functor from P-Alg to C. Let U' be the forgetful functor from P'-Alg to C. The composition of U and $-\circ\iota$ is natural isomorphic to U'. For a P-algebra (c, p), U' sends the parallel pair in Theorem 3.1 to

$$U'F'U\varepsilon_{(c,p)}=U'F'Up=U'F'p=P'p$$

and

$$\begin{aligned}
&U'\varepsilon'_{F'U(c,p)} \circ U'F'U\varepsilon_{(-\circ\iota)F'U(c,p)} \circ U'F'UF\eta'_{U(c,p)}\\
=&U'\varepsilon'_{F'U(c,p)} \circ U'F'U\varepsilon_{(P'c,\mu^{P'}_c\circ\iota_{P'c})} \circ U'F'UF\eta'_{U(c,p)}\\
=&\mu^{P'}_c \circ P'(\mu^{P'}_c \circ \iota_{P'c}) \circ P'P\eta'_c\\
=&\mu^{P'}_c \circ P'\mu^{P'}_c \circ P'P'\eta^{P'}_c \circ P'\iota_c\\
=&\mu^{P'}_c \circ P'\iota_c.
\end{aligned}$$

These pairs have an absolute coequalizer $e_{(c,p)}$. Since U' is monadic, $-\circ\iota$ has the left adjoint L by Theorem 3.2 and Theorem 3.1. □

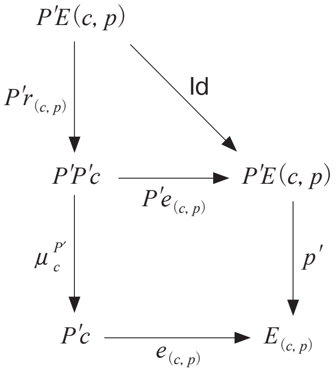
Theorem 3.5. *If the assumptions of Theorem 3.4 hold and for all P-algebra (c,p), $e_{(c,p)}$ has a right inverse $r_{(c,p)}$,*

$$\begin{array}{ccc}
E(c,p) & \xrightarrow{r_{(c,p)}} & P'c\\
& \searrow^{\mathrm{Id}} & \downarrow e_{(c,p)}\\
& & E(c,p)
\end{array}$$

then $L(c, p)$ is the pair of $E(c, p)$ and the following P'-structure map.

$$P'E(c,p) \xrightarrow{P'r_{(c,p)}} P'P'c \xrightarrow{\mu^{P'}_{c}} P'c \xrightarrow{e_{(c,p)}} E(c, p)$$

Proof. There exist a unique object $(E(c, p), p')$ and a unique arrow f: $(P'c, \mu^{P'}_{c}) \to (E(c, p), p')$ in P'-Alg satisfying $U'f = e_{(c, p)}$ by Theorem 3.4. Since f is $e_{(c, p)}$ itself and it is a P'-algebra homomorphism, the following diagram commutes.



□

By Theorem 3.4, the forgetful functor $-\circ\iota$ from **CSLat** to **SLat** has a left adjoint if **Set** has an absolute coequalizer of $\wp(\vee)$: $\wp(\wp_f(S)) \to \wp(S)$ and $\cup$: $\wp(\wp_f(S)) \to \wp(S)$ for a join semilattice $(S, \leq, \vee)$.

4　Ideal Completion as Absolute Coequalizer

This section shows that **Set** has an absolute coequalizer of $\wp(\vee)$ and $\cup$ for a join semilattice $(S, \leq, \vee)$.

Definition 4.1. Let S be a join semilattice. An *ideal* is a subset A of S such that

- A is closed under finite join operation $\vee$,
- A is closed downward under $\leq$.

Since an ideal A is closed under finite join, A must contain the least element $\vee\emptyset = 0$. Thus, ideals are not empty.

The set of ideals of a join semilattice S is denoted by $\mathcal{I}(S)$. For a subset A of a join semilattice S, we write $\langle A\rangle$ for

$\{a \in S \mid \exists X \in \wp_f(S).\ X \subseteq A,\ a \leq \bigvee X\}$.

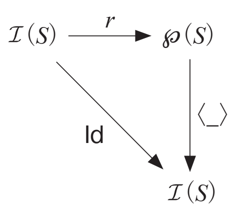
Lemma 4.2. *For a subset A and an ideal I of a join semilattice S, $\langle A\rangle$ is an ideal of S and $\langle A\rangle \subseteq I$ iff $A \subseteq I$. In other words, for a subset A of a join semilattice S, $\langle A\rangle$ is the smallest ideal containing A.*

Proof. For a finite subset $Y \subseteq \langle A\rangle$ and $y \in Y$, there exists a finite set X_y satisfying $X_y \subseteq A$ and $y \leq \vee X_y$. We have $\vee Y \in \langle A\rangle$, since $\cup\{X_y \mid y \in Y\}$ is a finite subset of S satisfying $\cup\{X_y \mid y \in Y\} \subseteq A$ and $\vee Y \leq \vee\cup\{X_y \mid y \in Y\}$. $\langle A\rangle$ is closed downward under $\leq$, trivially. Therefore, $\langle A\rangle$ is an ideal.

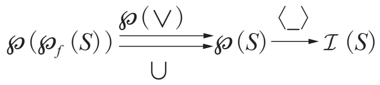
For all $a \in A$, $\{a\}$ is a finite subset satisfying $\{a\} \subseteq A$ and $a \leq \vee\{a\}$. Therefore, $\langle A\rangle$ contains A.

Let I be an ideal containing A. For $a \in S$ and $X \in \wp_f(S)$, if $X \subseteq A$ and $a \leq \vee X$, then we have $X \subseteq I$, $\vee X \in I$, and $a \in I$. Therefore, we have $\langle A\rangle \subseteq I$. □

The function $\langle\_\rangle$: $\wp(S) \to \mathcal{I}(S)$ which sends A to $\langle A\rangle$ is called an ideal completion. The inclusion function r: $\mathcal{I}(S) \to \wp(S)$ is a right inverse of $\langle\_\rangle$.



Theorem 4.3. *For a join semilattice S, $\langle\_\rangle: \wp(S) \to \mathcal{I}(S)$ is an absolute coequalizer of $\wp(\vee)$ and $\cup$ in* Set.



Proof. Let α be an element of $\wp(\wp_f(S))$. We have $\langle\wp(\vee)(\alpha)\rangle \subseteq \langle\cup\alpha\rangle$ as follows.

$\langle\wp(\vee)(\alpha)\rangle \subseteq \langle\cup\alpha\rangle$
$\Leftrightarrow \wp(\vee)(\alpha) \subseteq \langle\cup\alpha\rangle$ (by Lemma 4.2)
$\Leftrightarrow \vee X \in \langle\cup\alpha\rangle$ $(\forall X \in \alpha)$
$\Leftrightarrow X \subseteq \langle\cup\alpha\rangle$ $(\forall X \in \alpha)$ (since an ideal is closed under finite join)
$\Leftrightarrow \cup\alpha \subseteq \langle\cup\alpha\rangle$
$\Leftrightarrow \langle\cup\alpha\rangle \subseteq \langle\cup\alpha\rangle$ (by Lemma 4.2)

Conversely, we have $\langle\cup\alpha\rangle \subseteq \langle\wp(\vee)(\alpha)\rangle$ as follows.

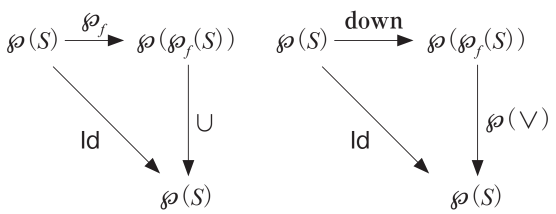
$\langle\cup\alpha\rangle \subseteq \langle\wp(\vee)(\alpha)\rangle$
$\Leftrightarrow \cup\alpha \subseteq \langle\wp(\vee)(\alpha)\rangle$ (by Lemma 4.2)
$\Leftrightarrow X \subseteq \langle\wp(\vee)(\alpha)\rangle$ $(\forall X \in \alpha)$
$\Leftrightarrow \vee X \in \langle\wp(\vee)(\alpha)\rangle$ $(\forall X \in \alpha)$ (since an ideal is closed downward)
$\Leftrightarrow \wp(\vee)(\alpha) \subseteq \langle\wp(\vee)(\alpha)\rangle$
$\Leftrightarrow \langle\wp(\vee)(\alpha)\rangle \subseteq \langle\wp(\vee)(\alpha)\rangle$ (by Lemma 4.2)

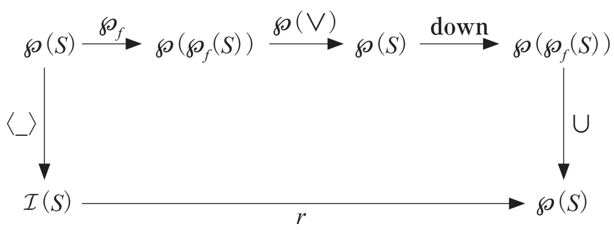
Therefore, for each $\alpha \in \wp(\wp_f(S))$, $\langle\wp(\vee)(\alpha)\rangle = \langle\cup\alpha\rangle$.

We define $\wp_f: \wp(S) \to \wp(\wp_f(S))$ and **down**: $\wp(S) \to \wp(\wp_f(S))$ as follows.

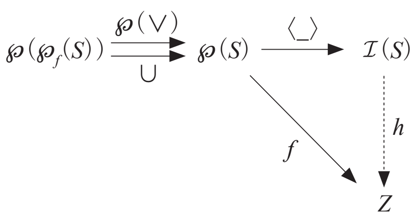
$\wp_f(A) = \{X \in \wp_f(S) \mid X \subseteq A\}$
$\mathbf{down}(A) = \{\{a,b\} \mid a \in S, b \in A, a \leq b\}$

These functions satisfy the following diagrams, where r is the inclusion function from $\mathcal{I}(S)$ to $\wp(S)$.





Let $Z \in$ Set and f: $\wp(S) \to Z$ be satisfying $f \circ \wp(\vee) = f \circ \cup$. We show that there exists a unique arrow h: $\mathcal{I}(S) \to Z$ satisfying $h \circ \langle\_\rangle = f$.



The arrow h is defined by $f \circ r$. We have $h \circ \langle\_\rangle = f$ as follows:

$$\begin{aligned} h \circ \langle\_\rangle &= f \circ r \circ \langle\_\rangle \\ &= f \circ \cup \circ \mathbf{down} \circ \wp(\vee) \circ \wp_f \\ &= f \circ \wp(\vee) \circ \mathbf{down} \circ \wp(\vee) \circ \wp_f \\ &= f \circ \wp(\vee) \circ \wp_f \\ &= f \circ \cup \circ \wp_f \\ &= f \end{aligned}$$

Moreover, if an arrow g: $\mathcal{I}(S) \to Z$ satisfies $g \circ \langle\_\rangle = f$, then $h = f \circ r = g \circ \langle\_\rangle \circ r = g$. So, h: $\mathcal{I}(S) \to Z$ is a unique arrow satisfying $h \circ \langle\_\rangle = f$.

Therefore, $\langle\_\rangle$: $\wp(S) \to \mathcal{I}(S)$ is a coequalizer of $\wp(\vee)$ and $\cup$ in Set. The above diagrams are preserved by any functor from Set to another category. Therefore, $\langle\_\rangle$: $\wp(S) \to \mathcal{I}(S)$ is an absolute coequalizer of $\wp(\vee)$ and $\cup$ in Set.

□

Example 4.4. By Theorem 4.3, Theorem 3.4, and Theorem 3.5, the forgetful functor from CSLat to SLat has a left adjoint, which sends a join semilattice S to $(\mathcal{I}(S), \subseteq, \vee)$ satisfying $\vee\alpha = \langle \cup \alpha \rangle$.

5 Left Adjoint between Categories of Algebras by Absolute Coequalizers

This section extends Theorem 3.4 to the theorem for quantales and idempotent semirings.

Definition 5.1. An *idempotent semiring*, abbreviated as *I-semiring* is a tuple $(S, +, \cdot, 0, 1)$ with a set S, two binary operations + and ・, and $0, 1 \in S$ satisfying the following properties:

- $(S, +, 0)$ is an idempotent commutative monoid.
- $(S, \cdot, 1)$ is a monoid.
- For all $a, b, c \in S$,

$$a \cdot c + b \cdot c = (a+b) \cdot c$$
$$a \cdot b + a \cdot c = a \cdot (b+c)$$
$$0 \cdot a = 0$$
$$a \cdot 0 = 0$$

where the *natural order* $\leq$ is given by $a \leq b$ iff $a+b=b$.

We often abbreviate $a \cdot b$ to ab.

The natural order $\leq$ on an I-semiring is a join semilattice, where its join operation is given by $\vee\emptyset = 0$ and, for a finite subset $A \subseteq S$ containing a, $\vee A = a + (\vee A \setminus \{a\})$.

Example 5.2. Let Σ be a finite set and Σ^* the set of finite words (strings) over Σ. Then, the finite power set $\wp_f(\Sigma^*)$ of Σ^* forms an I-semiring together with the union, concatenation, empty set, and the singleton set of the empty word.

IS denotes the category whose objects are I-semirings and whose arrows are homomorphisms between them.

Definition 5.3. A *quantale* S is an I-semiring satisfying the following properties: For each $A \subseteq S$ and $a \in S$,

- the least upper bound $\bigvee A$ of A exists in S,
- $(\bigvee A)a = \bigvee \{xa \mid x \in A\}$, and
- $a(\bigvee A) = \bigvee \{ax \mid x \in A\}$.

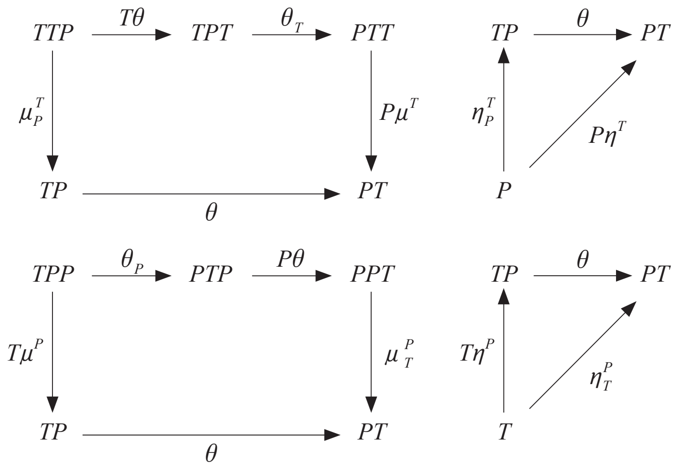
So, a quantale is a complete I-semiring or an S-algebra [3]. Homomorphisms between quantales are semiring homomorphisms preserving arbitrary joins.

Example 5.4. Let Σ be a finite set and Σ^* the set of finite words (strings) over Σ. Then, the power set $\wp(\Sigma^*)$ of Σ^* forms a quantale together with the union, concatenation, empty set, and the singleton set of the empty word.

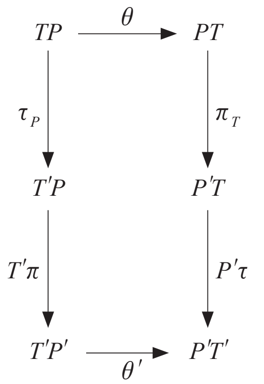
Qt denotes the category whose objects are quantales and whose arrows are homomorphisms between them.

Remark 5.5. I-semirings need not be quantales. For example, an I-semiring $\wp_f(\Sigma^*)$ is not a quantale since it is not closed under arbitrary unions.

Definition 5.6 (distributive law [7]). Let $T=(T, \mu^T, \eta^T)$ and $P=(P, \mu^P, \eta^P)$ be monads on a category C. A *distributive law* θ of P over T is a natural transformation from TP to PT satisfying the following diagrams.

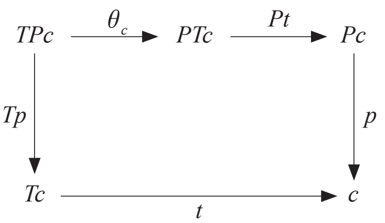


Definition 5.7. Let T, T', P, P' be monads on C. Let θ be a distributive law of P over T. Let θ' be a distributive law of P' over T'. A *morphism* (τ, π): $\theta \rightarrow \theta'$ *of distributive laws* consists of monad maps τ: $T \rightarrow T'$ and π: $P \rightarrow P'$ satisfying the following diagram.



Definition 5.8. Let P and T be monads on a category C. Let θ be a distributive law of P over T. A $P\circ_\theta$ *T-algebra* is a tuple (c, t, p) such that

- c is an object in C,
- The pair of c and t: $Tc \rightarrow c$ is a T-algebra,
- The pair of c and p: $Pc \rightarrow c$ is a P-algebra, and
- $p \circ Pt \circ \theta_c = t \circ Tp$.



$P \circ_\theta T$-Alg denotes the category whose objects are $P \circ_\theta T$-algebras and whose arrows are simultaneous T- and P-homomorphisms.

Lemma 5.9. *Let T, P, P' be monads on C. Let θ be a distributive law of P over T. Let θ' be a distributive law of P' over T. Let ι be a monad map such that* (Id, ι) is a morphism (Id, ι): *$\theta \rightarrow \theta'$ of distributive laws. The following G is a functor from $P' \circ_{\theta'} T$-Alg to $P \circ_\theta T$-Alg.*

- *For a $P'\circ_{\theta'}$ T-algebra* (c, t, p'), $G(c, t, p') = (c, t, p'\circ\iota_c)$.

・*For a* $P'\circ_{\theta'}$ *T-algebra homomorphism f,* $Gf = f$.

We write $-\circ\iota$ for the above functor G.

Example 5.10. Let $T=(\_)^*$ be the monad for finite sequences on Set. Then, T-Alg is equivalent to the category Mon whose objects are monoids and whose arrows are homomorphisms between them.

Let $\wp_f$ be the finite powerset monad $(\wp_f, \cup, \{\_\})$ on Set. Let $\wp$ be the powerset monad $(\wp, \cup, \{\_\})$ on Set. There exists a distributive law θ of T over $\wp_f$, and there exists a distributive law θ' of T over $\wp$ as follows.

$$\theta_X(S_1 \cdot S_2 \cdots\cdots S_n) = \{x_1 \cdot x_2 \cdots\cdots x_n \mid x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n\}$$

$$\theta'_X(S_1 \cdot S_2 \cdots\cdots S_n) = \{x_1 \cdot x_2 \cdots\cdots x_n \mid x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n\}$$

Then, $\wp_f \circ_\theta T$-Alg is equivalent to the category IS and $\wp\circ_{\theta'} T$-Alg is equivalent to the category Qt. Let ι_X be the inclusion function from $\wp_f(X)$ to $\wp(X)$. $(\mathsf{Id}, \iota): \theta \to \theta'$ is a morphism of distributive laws.

Lemma 5.11. *The forgetful functor from* $P\circ_\theta$ *T-Alg to C is monadic.*

Example 5.12. The forgetful functor from Mon to Set, the forgetful functor from IS to Set, and the forgetful functor from Qt to Set are monadic.

Lemma 5.13. *The forgetful functor from* $P\circ_\theta$ *T-Alg to T-Alg is monadic. The left adjoint to this forgetful functor sends a T-algebra* (c, t) *to* $(Pc, Pt\circ\theta_c, \mu^P_c)$ *and a T-homomorphism f to Pf. The unit for a T-algebra* (c, t) *is* η^P_c: $(c, t) \to (Pc, Pt\circ\theta_c)$. *The counit for a* $P\circ_\theta$ *T-algebra* (c, t, p) *is p:* $(Pc, Pt \circ \theta_c, \mu^P_c) \to (c, t, p)$.

Example 5.14. The forgetful functor from IS to Mon and the forgetful functor from Qt to Mon are monadic.

Theorem 5.15. *Let* $T=(T, \mu^T, \eta^T)$, $P=(P, \mu^P, \eta^P)$, *and* $P'=(P', \mu^{P'}, \eta^{P'})$ *be monads on C. Let θ be a distributive law of P over T. Let θ′ be a distributive law of P′ over T. Let ι be a monad map such that* (Id, ι) *is a morphism* $(\mathsf{Id}, \iota): \theta \to \theta'$ *of distributive laws. For a* $P\circ_\theta$ *T-algebra* (c, t, p), *let* $e_{(c,p)}: P'c \to E(c, p)$ *be an absolute coequalizer of* $P'p$ *and* $\mu^{P'}_c \circ P'\iota_c$ *in C.*

$$P'Pc \underset{\mu^{P'}_c \circ P'\iota_c}{\overset{P'p}{\rightrightarrows}} P'c \xrightarrow{e_{(c,p)}} E(c,p)$$

Then, the functor $-\circ\iota$: $P'\circ_{\theta'}$ *T-Alg* $\to P\circ_\theta$ *T-Alg has a left adjoint L, where* $L(c, t, p)$ *is the* $P'\circ_{\theta'}$ *T-algebra on* $E(c, p)$ *created by the forgetful functor from* $P'\circ_{\theta'}$ *T-Alg to C.*

Proof. Let $D=P\circ_\theta T$-Alg and $D'=P'\circ_{\theta'} T$-Alg in Theorem 3.1. Let U be the forgetful functor from $P\circ_\theta T$-Alg to T-Alg. Let U' be the forgetful functor from $P'\circ_{\theta'} T$-Alg to T-Alg. The composition of U and $-\circ\iota$ is natural isomorphic to U'. For a $P\circ_\theta T$-algebra (c, t, p), the forgetful functor U' sends the parallel pair in Theorem 3.1 to

$$U'F'U\varepsilon_{(c,t,p)}=U'F'Up=U'F'p=P'p$$

and

$$\begin{aligned}
&U'\varepsilon'_{F'U(c,t,p)}\circ U'F'U\varepsilon_{(-\circ\iota)F'U(c,t,p)} \circ U'F'UF\eta'_{U(c,t,p)}\\
&=U'\varepsilon'_{F'U(c,t,p)} \circ U'F'U\varepsilon_{(P'c,\, P't\circ\theta'_c,\, \mu^{P'}_c\circ\iota_{P'c})}\circ U'F'UF\eta'_{U(c,t,p)}\\
&=\mu^{P'}_c \circ P'(\mu^{P'}_c\circ\iota_{P'c})\circ P'P\eta'_c\\
&=\mu^{P'}_c \circ P'\mu^{P'}_c\circ P'P'\eta^{P'}_c\circ P'\iota_c\\
&=\mu^{P'}_c \circ P'\iota_c
\end{aligned}$$

Moreover, the forgetful functor from T-Alg to C sends these pairs to the same arrows. They have an absolute coequalizer. The monadic forgetful functor from $P'\circ_{\theta'} T$-Alg to C is monadic. Therefore, $-\circ\iota$ has the left adjoint L by Theorem 3.1 and Theorem 3.2. □

Theorem 5.16. *If the assumptions of Theorem 5.15 hold and for all* $P\circ_\theta$ *T-algebra* (c, t, p), $e_{(c,p)}$ *has a right inverse* $r_{(c,p)}$,

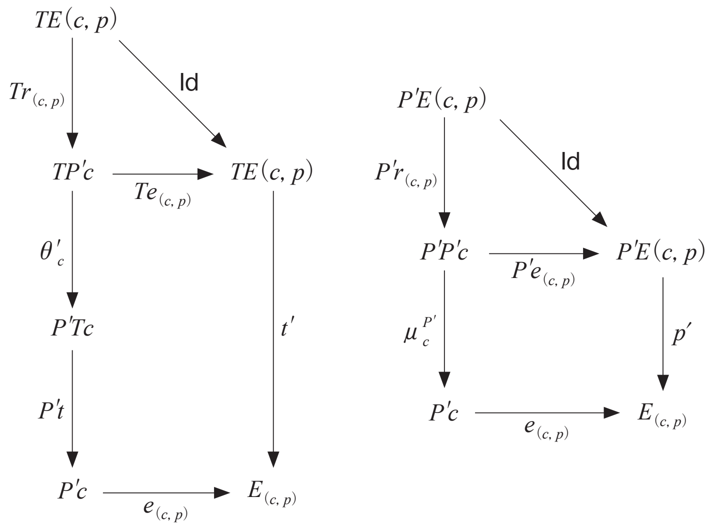
$$\begin{array}{ccc}
E(c,p) & \xrightarrow{r_{(c,p)}} & P'c \\
 & {\scriptstyle \mathsf{Id}}\searrow & \downarrow{\scriptstyle e_{(c,p)}} \\
 & & E(c,p)
\end{array}$$

then $L(c, p)$ is the tuple of $E(c, p)$ and the following $P' \circ_{\theta'} T$-structure map.

$$TE(c, p) \xrightarrow{Tr_{(c,p)}} TP'c \xrightarrow{\theta'_c} P'Tc \xrightarrow{P't} P'c \xrightarrow{e_{(c,p)}} E(c, p)$$

$$P'E(c, p) \xrightarrow{Pr_{(c,p)}} P'P'c \xrightarrow{\mu^{P'}_c} P'c \xrightarrow{e_{(c,p)}} E(c, p)$$

Proof. There exist a unique object $(E(c, p), t', p')$ and a unique arrow f: $(P'c, P't \circ \theta'_c, \mu^{P'}_c) \to (E(c, p), t', p')$ in $P'_{\theta'}$ T-Alg satisfying $U'f = e_{(c, p)}$ by Theorem 5.15. Since f is $e_{(c, p)}$ itself and it is simultaneous a T-homomorphism and a P'-homomorphism, both of the following diagrams commute.



□

Example 5.17. By Theorem 5.15, Theorem 5.16, and Theorem 4.3, the forgetful functor from Qt to IS has a left adjoint, which sends an idempotent semiring $(S, +, \cdot, 0, 1)$ to $(\mathcal{I}(S), \subseteq, \vee_I, \cdot_I, 1_I)$ satisfying

$\bigvee_I \alpha = \langle \bigcup \alpha \rangle$, $J \cdot_I K = \langle \{a \cdot b \mid a \in J, b \in K\} \rangle$, $1_I = \langle \{1\} \rangle$.

6 Direct Proof of Left Adjoint by Ideal Completion

In this section, we directly prove that the ideal completion gives the left adjoint to the forgetful functor from Qt to IS.

Definition 6.1 (cf. Definition 4.1). Let S be an I-semiring. An *ideal* is a subset A of S such that A is an ideal of the underlying join semilattice $(S, \leq, \vee)$.

The set of ideals of an I-semiring S is denoted by $\mathcal{I}(S)$. Note that $\mathcal{I}(S)$ is closed under arbitrary intersection. Also note that $A \in \mathcal{I}(S)$ iff A is nonempty, closed under +, and closed downward under ≤.

We say that a subset A of an I-semiring S *generates* an ideal I if I is the smallest ideal containing A. By Lemma 4.2, $\langle A \rangle = \{a \in S \mid \exists X \in \wp_f(S).\ X \subseteq A,\ a \leq \vee X\}$ is the ideal generated by $A \subseteq S$. Note that $\langle \_ \rangle$ is monotone and idempotent, i. e. $A \subseteq B$ implies $\langle A \rangle \subseteq \langle B \rangle$ and $\langle \langle A \rangle \rangle = \langle A \rangle$ for any subsets $A, B \subseteq S$. If A is a singleton $\{a\}$, we often abbreviate $\langle \{a\} \rangle$ to $\langle a \rangle$. Such an ideal is called *principal*.

Lemma 6.2. *Let S be an I-semiring. For a subset α of $\wp(S)$, $\langle \cup \alpha \rangle = \langle \cup \{\langle A \rangle \mid A \in \alpha\} \rangle$.*

Proof. The inclusion ⊆ follows from monotonicity of $\langle \_ \rangle$. Again, by monotonicity of $\langle \_ \rangle$, $\langle A \rangle \subseteq \langle \bigcup \alpha \rangle$ for each $A \in \alpha$. Thus, $\bigcup \{\langle A \rangle \mid A \in \alpha\} \subseteq \langle \bigcup \alpha \rangle$. So, we have $\langle \bigcup \{\langle A \rangle \mid A \in \alpha\} \rangle \subseteq \langle \bigcup \alpha \rangle$ by monotonicity and idempotency of $\langle \_ \rangle$. □

Let S be an I-semiring. For subsets $A, B \subseteq S$, define

$A \oplus B = \{a + b \mid a \in A, b \in B\}$

$A \odot B = \{a \cdot b \mid a \in A, b \in B\}$

$A \downarrow = \{y \mid \exists x \in A.\ y \leq x\}$.

Note that

$(A \oplus B) \odot C \subseteq (A \odot C) \oplus (B \odot C)$
$C \odot (A \oplus B) \subseteq (C \odot A) \oplus (C \odot B)$
$A \odot (B\downarrow) \subseteq (A \odot B)\downarrow$
$(A\downarrow) \odot B \subseteq (A \odot B)\downarrow.$

Also note that for principal ideals,
$\langle a \rangle = \{a\}\downarrow.$

Though Lemma 3 in [4], corresponding property will appear as Lemma 6.5 in this paper, is shown by transfinite induction, we are going to show Lemma 6.5 without using transfinite induction.

For subsets A, B, X of an I-semiring S, define
$x \in A^{-1}X \Leftrightarrow \forall a \in A.\ ax \in X$
$x \in XB^{-1} \Leftrightarrow \forall b \in B.\ xb \in X.$

Note that $B \subseteq A^{-1}X$ iff $A \odot B \subseteq X$ iff $A \subseteq XB^{-1}$. Then, since $A^{-1}X \subseteq A^{-1}X$ and $XB^{-1} \subseteq XB^{-1}$, we have $A \odot A^{-1}X \subseteq X$ and $XB^{-1} \odot B \subseteq X$. Also, if $X \subseteq Y$, $A \subseteq A'$, $B \subseteq B'$, it holds that $A'^{-1}X \subseteq A^{-1}Y$ and $XB'^{-1} \subseteq YB^{-1}$.

Lemma 6.3. *The following holds for subsets X, Y, A and B of an I-semiring S.*

1. $A^{-1}X \oplus A^{-1}Y \subseteq A^{-1}(X \oplus Y)$.
2. $XB^{-1} \oplus YB^{-1} \subseteq (X \oplus Y)B^{-1}$.
3. $(A^{-1}X)\downarrow \subseteq A^{-1}(X\downarrow)$.
4. $(XB^{-1})\downarrow \subseteq (X\downarrow)B^{-1}$.

Proof. 1 follows from $A \odot (A^{-1}X \oplus A^{-1}Y) \subseteq (A \odot A^{-1}X) \oplus (A \odot A^{-1}Y) \subseteq X \oplus Y$. The other inclusions are shown similarly to 1. □

Lemma 6.4. *Let S be an I-semiring. The following holds for an ideal I and subsets A, B ⊆ S.*

1. *$A^{-1}I$ is an ideal.*
2. *IB^{-1} is an ideal.*

Proof. 1. Since $A \odot \{0\} = \{0\} \subseteq I$, $\{0\} \subseteq A^{-1}I$. Thus, $A^{-1}I$ is nonempty. Also, we have

$$A^{-1}I \oplus A^{-1}I \subseteq A^{-1}(I \oplus I) \quad \text{(by 1 of Lemma 6.3)}$$
$$\subseteq A^{-1}I,$$
$$(A^{-1}I)\downarrow \subseteq A^{-1}(I\downarrow) \quad \text{(by 3 of Lemma 6.3)}$$
$$\subseteq A^{-1}I.$$

Therefore $A^{-1}I$ is an ideal. 2 is proved similarly to 1. □

Lemma 6.5. *Let S be an I-semiring. Then, $\langle A \odot B \rangle = \langle \langle A \rangle \odot \langle B \rangle \rangle$ holds for any subsets $A, B \subseteq S$.*

Proof. The inclusion ⊆ follows from monotonicity of ⟨_⟩. Next, we show the reverse inclusion. Since $A \odot B \subseteq \langle A \odot B \rangle$, we have $A \subseteq \langle A \odot B \rangle B^{-1}$. By Lemma 6.4 and monotonicity of ⟨_⟩, it holds that $\langle A \rangle \subseteq \langle A \odot B \rangle B^{-1}$. Thus, $\langle A \rangle \odot B \subseteq \langle A \odot B \rangle$ holds. Moreover, this inclusion implies $B \subseteq \langle A \rangle^{-1} \langle A \odot B \rangle$. Again, by Lemma 6.4 and monotonicity of ⟨_⟩, $\langle B \rangle \subseteq \langle A \rangle^{-1} \langle A \odot B \rangle$. Thus, $\langle A \rangle \odot \langle B \rangle \subseteq \langle A \odot B \rangle$ holds. Therefore, by monotonicity of ⟨_⟩, the reverse inclusion holds. □

In general, the natural ordering on an I-semiring is a join semilattice but not complete as we have seen in Remark 5.5. Next, we provide a completion of I-semirings.

Let S be an I-semiring and consider the poset $(\mathcal{I}(S), \subseteq)$. Since $\langle 0 \rangle = \{0\}$, $\langle 0 \rangle$ is the least element of $\mathcal{I}(S)$. For each subset $\alpha \subseteq \mathcal{I}(S)$, an ideal $\langle \bigcup \alpha \rangle$ is the least upper bound of α. Obviously, the least upper bound of $\emptyset \subseteq \mathcal{I}(S)$ is $\langle 0 \rangle$. Also, since $\langle a \rangle = \{a\}\downarrow$,
$a \leq b \Leftrightarrow \langle a \rangle \subseteq \langle b \rangle$
holds for any $a, b \in S$. Thus, $(\mathcal{I}(S), \subseteq)$ is an ideal completion of $(S, \leq)$ with an embedding $\langle \_ \rangle: S \to \mathcal{I}(S)$. $\bigvee \alpha$ denotes the least upper bound of $\alpha \subseteq \mathcal{I}(S)$. For any $I, J \in \mathcal{I}(S)$, we write $I+J$ for $\bigvee \{I, J\}$, and define $I \cdot J = \langle I \odot J \rangle$.

In the rest of this section, a functor from IS to Qt which is a left adjoint to the forgetful functor is provided. G: Qt → IS

Proposition 6.6. *Let S be an I-semiring. For any $H, I, J \in \mathcal{I}(S)$ and $\alpha \subseteq \mathcal{I}(S)$, the following holds.*

1. $\langle 0\rangle \cdot I=\langle 0\rangle=I\cdot\langle 0\rangle$.
2. $\langle 1\rangle \cdot I=I=I\cdot\langle 1\rangle$.
3. $(H\cdot I)\cdot J=H\cdot(I\cdot J)$.
4. $(\bigvee\alpha)\cdot I=\bigvee\{J\cdot I \mid J\in\alpha\}$.
5. $I\cdot(\bigvee\alpha)=\bigvee\{I\cdot J \mid J\in\alpha\}$.

Proof. 1 follows from definition of $\odot$. 2 and 3 follow from definition of $\odot$ and Lemma 6.5. 4 follows from

$$\begin{aligned}(\bigvee\alpha)\cdot I&=\langle(\bigvee\alpha)\odot I\rangle\\ &=\langle\langle\bigcup\alpha\rangle\odot I\rangle\\ &=\langle(\bigcup\alpha)\odot I\rangle &&\text{(by Lemma 6.5)}\\ &=\langle\bigcup\{J\odot I \mid J\in\alpha\}\rangle\\ &=\langle\bigcup\{\langle J\odot I\rangle \mid J\in\alpha\}\rangle &&\text{(by Lemma 6.2)}\\ &=\langle\bigcup\{J\cdot I \mid J\in\alpha\}\rangle\\ &=\bigvee\{J\cdot I \mid J\in\alpha\}.\end{aligned}$$

5 is proved similarly to 4. □

Therefore, $\mathcal{I}(S)$ forms a quantale.

Let S be an I-semiring. Using $\langle x\rangle=\{x\}\downarrow$, it is verified that the mapping

$$x\mapsto\langle x\rangle$$

from S to $\mathcal{I}(S)$ is one-to-one and preserves $+$, $\cdot$, 0, and 1. Thus, this mapping is an arrow from S to $\mathcal{I}(S)$ in **IS**.

We have omitted a transfinitely inductive construction of an ideal, which has been adopted in [4]. Thus, it is impossible to benefit from the proof of Lemma 4 in [4], which depends on the transfinitely inductive construction.

Let S and S' be I-semirings. Given a homomorphism f from S to S', we define

$$f[A]=\{f(a) \mid a\in A\} \text{ and } f^{-1}[A']=\{a \mid f(a)\in A'\}$$

for each $A\subseteq S$ and $A'\subseteq S'$, respectively. Note that $f[A]\subseteq A'$ iff $A\subseteq f^{-1}[A']$. Also, note that

$$f[A\odot B]=f[A]\odot f[B]$$
$$f[A\oplus B]=f[A]\oplus f[B]$$
$$f[A\downarrow]\subseteq f[A]\downarrow$$

for all $A, B\subseteq S$.

Lemma 6.7. *Let S and S' be I-semirings and $f: S\to S'$ a homomorphism. Then the following holds for any $A', B'\subseteq S'$.*

1. $f^{-1}[A']\odot f^{-1}[B']\subseteq f^{-1}[A'\odot B']$.
2. $f^{-1}[A']\oplus f^{-1}[B']\subseteq f^{-1}[A'\oplus B']$.
3. $f^{-1}[A']\downarrow\subseteq f^{-1}[A'\downarrow]$.
4. *If $A'\in\mathcal{I}(S')$, then $f^{-1}[A']\in\mathcal{I}(S)$.*

Proof. The first inclusion is equivalent to $f[f^{-1}[A']\odot f^{-1}[B']]\subseteq A'\odot B'$. Also, it holds that

$$f[f^{-1}[A']\odot f^{-1}[B']]=f[f^{-1}[A']]\odot f[f^{-1}[B']].$$

Since $f[f^{-1}[A']]\subseteq A'$ and $f[f^{-1}[B']]\subseteq B'$, 1 holds. 2 is proved similarly to 1. The third inclusion is equivalent to $f[f^{-1}[A']\downarrow]\subseteq A'\downarrow$. Since $f[f^{-1}[A']\downarrow]\subseteq f[f^{-1}[A']]\downarrow$ and $f[f^{-1}[A']]\subseteq A'$, 3 holds. It is sufficient for 4 to check nonemptiness of $f^{-1}[A']$ since the others are induced by 2 and 3. Suppose that A' is an ideal. Then, $0\in A'$. So, $f[\{0\}]=\{0\}\subseteq A'$. Thus $\{0\}\subseteq f^{-1}[A']$. Therefore, $f^{-1}[A']$ is nonempty. □

Lemma 6.8. *Let S and S' be I-semirings and $f: S\to S'$ a homomorphism. For each subset $A\subseteq S$, the following holds.*

$$f[\langle A\rangle]\subseteq\langle f[A]\rangle$$
$$\langle f[\langle A\rangle]\rangle=\langle f[A]\rangle$$

Proof. The first inclusion follows from

$$\begin{aligned}f[A]\subseteq\langle f[A]\rangle&\Leftrightarrow A\subseteq f^{-1}[\langle f[A]\rangle]\\ &\Rightarrow\langle A\rangle\subseteq f^{-1}[\langle f[A]\rangle] &&\text{(by 4 of Lemma 6.7)}\\ &\Leftrightarrow f[\langle A\rangle]\subseteq\langle f[A]\rangle.\end{aligned}$$

The inclusion $\langle f[\langle A\rangle]\rangle \subseteq \langle f[A]\rangle$ follows from the first, and the reverse inclusion follows from the monotonicity of f and $\langle\_\rangle$. □

Let S be an I-semiring, Q a quantale, and g: $S \to G(Q)$ a homomorphism. By Lemma 6.8, $g[I]$ and $g[A]$ generate the same ideal if $I \in \mathcal{I}(S)$ is generated by $A \subseteq S$. For $I \in \mathcal{I}(S)$, the least upper bound of $g[I]$ exists in Q, which is denoted by $\bigvee g[I]$, since the least upper bound of any subset of Q exists in Q.

Lemma 6.9. *Let S be an I-semiring, Q a quantale, and g: $S \to G(Q)$ a homomorphism. If $I \in \mathcal{I}(S)$, $\bigvee g[I] = \bigvee g[A]$ for any generating set A of I.*

Proof. By Lemma 6.8, $\langle g[I]\rangle = \langle g[\langle A\rangle]\rangle = \langle g[A]\rangle$ for any generating set A of I. Then, $\bigvee g[A]$ is an upper bound of $g[I]$ since $\langle g[I]\rangle = \langle g[A]\rangle \subseteq \langle \bigvee g[A]\rangle = (\bigvee g[A])\downarrow$. By $g[A] \subseteq g[I]$, $\bigvee g[A] \le \bigvee g[I]$. Thus, $\bigvee g[A] = \bigvee g[I]$ since $\bigvee g[I]$ is the least upper bound of $g[I]$. □

Let S be an I-semiring and Q a quantale. Define the map $\hat{g}$: $\mathcal{I}(S) \to Q$ by

$$\hat{g}(I) = \bigvee g[I]$$

for a homomorphism g: $S \to G(Q)$.

Proposition 6.10. *The map $\hat{g}$ preserves $\bigvee$, $\cdot$, 0 and 1.*

Proof. For $I, J \in \mathcal{I}(S)$,

$$\begin{aligned}\hat{g}(I\cdot J) &= \bigvee g[I\cdot J]\\ &= \bigvee g[\langle I \odot J\rangle]\\ &= \bigvee g[I \odot J] && \text{(by Lemma 6.9)}\\ &= \bigvee g[I] \odot g[J]\\ &= (\bigvee g[I])\cdot(\bigvee g[J])\\ &= \hat{g}(I)\cdot\hat{g}(J).\end{aligned}$$

Also, we have $\hat{g}(\langle 1\rangle) = \bigvee g[\langle 1\rangle] = \bigvee g[\{1\}] = 1$ and $\hat{g}(\langle 0\rangle) = \bigvee g[\langle 0\rangle] = \bigvee g[\{0\}] = 0$. Note that $\bigvee(\bigcup \beta) = \bigvee\{\bigvee B \mid B \in \beta\}$ for a subset β of the powerset $\wp(Q)$. Then, for nonempty subset $\alpha \subseteq \mathcal{I}(S)$,

$$\begin{aligned}\hat{g}(\bigvee\alpha) &= \bigvee g[\bigvee\alpha]\\ &= \bigvee g[\langle \bigcup\alpha\rangle]\\ &= \bigvee g[\bigcup\alpha] && \text{(by Lemma 6.9)}\\ &= \bigvee(\bigcup\{g[I] \mid I\in\alpha\})\\ &= \bigvee\{\bigvee g[I] \mid I\in\alpha\}\\ &= \bigvee\{\hat{g}[I] \mid I\in\alpha\}.\end{aligned}$$

The equation holds even in the case of $\alpha = \emptyset$ since $\bigvee\alpha = \langle 0\rangle$ and it has been shown that $\hat{g}$ preserves 0. □

Theorem 6.11. *Let S be an I-semiring and Q a quantale. For a homomorphism g: $S \to G(Q)$, $\hat{g}$ is a unique completely join-preserving homomorphism from $\mathcal{I}(S)$ to Q such that $g = \hat{g} \circ \langle\_\rangle$.*

Proof. For each $a \in S$, we have $\hat{g}(\langle a\rangle) = \bigvee g[\langle a\rangle] = \bigvee g[\{a\}] = g(a)$ by Lemma 6.9. Assume that a completely join-preserving homomorphism f from $\mathcal{I}(S)$ to Q satisfies $g = f \circ \langle\_\rangle$. Then, it holds that

$$\begin{aligned}\hat{g}(I) &= \bigvee g[I]\\ &= \bigvee\{g(\alpha) \mid \alpha \in I\}\\ &= \bigvee\{f(\langle\alpha\rangle) \mid \alpha\in I\} && \text{(by assumption)}\\ &= f(\bigvee\{\langle\alpha\rangle \mid \alpha\in I\})\\ &= f(I)\end{aligned}$$

for each $I \in \mathcal{I}(S)$. Thus, $\hat{g} = f$. □

For an I-semiring S and a homomorphism h: $S \to S'$, we define

$$F(S) = \mathcal{I}(S) \text{ and } F(h) = \widehat{\langle\_\rangle \circ h},$$

respectively. Then, F is a functor from IS to Qt. It is immediate from Theorem 6.11 that the following holds.

Corollary 6.12. *The functor F: $\mathsf{IS} \to \mathsf{Qt}$ is a left adjoint to the forgetful functor G: $\mathsf{Qt} \to \mathsf{IS}$.*

7　Conclusion and Future Work

We provided the sufficient condition for the functor from $P' \circ_{\theta'} T$-Alg to $P \circ_{\theta} T$-Alg to have a left adjoint. This result includes the cases of the forgetful functor from CSLat to SLat and the forgetful functor from Qt to IS. For the second case, a proof being independent from the sufficient condition was also provided.

In both cases, left adjoints are given by an ideal completion. The authors plan to search for other examples than ideal completion provided.

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References

[1]　Barr M., Wells, C.: Category Theory for Computing Science, Third Edition Les Publications (1999).

[2]　Barr, M., Wells, C.: Toposes, Triples and Theories Volume 278 of Grundlagen der Mathematischen Wissenschaften. Springer-Verlag, Berlin (1985).

[3]　Conway, J.H.: *Regular Algebra and Finite Machines* Chapman and Hall (1971).

[4]　Kozen, D.: On Kleene Algebras and Closed Semirings In: Rovan, B. (ed.) MFCS 1990. LNCS, vol. 452, pp. 26-47. Springer (1990).

[5]　Kozen, D.: A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events *Information and Computation*, 110: 366-390 (1994).

[6]　Mac Lane, S.: Categories for the Working Mathematician, Second Edition Springer (1998).

[7]　Manes, E., and Mulry, P.: Monad compositions I: general constructions and recursive distributive laws. Theory and Applications of Categories 18: 172-208 (2007).

[8]　Mulvey, C.J.: “&”, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II no. 12, pp. 99-104 (1986).

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